l.a) Particle on a sphere

\[ E = \frac{l(l+1) \hbar^2}{2I} = 2.695 \times 10^{-20} \text{ J} \]

Approach: evaluate \( l \) and determine if \( l \) is an integer.

\[ l(l+1) = \frac{2IE}{\hbar^2} = \frac{2(1.448 \times 10^{-46} \text{ kg m}^2)(2.695 \times 10^{-20} \text{ kg m}^2)}{\left(\frac{1}{2n}\right)^2 (6.626 \times 10^{-34} \text{ kg m}^2)^2} \]

\[ l(l+1) = 701.8 \]

\[ l^2 + l - 701.8 = 0 \]

\[ l = \frac{-1 \pm \sqrt{1^2 - 4(1)(-701.8)}}{2} = -1 \pm 53 \]

\[ l = 26 \text{ or } -54 \]

allowed state of a particle on a sphere

Angular momentum

\[ L = \sqrt{l(l+1)} \hbar = \sqrt{26(27)} \hbar = \sqrt{702} \hbar \]

26.50 \( \hbar \) is also OK

Degeneracy comes from allowed \( m_l \) values

\[ m_l = -l, -l+1, \ldots, l \]

2 \( l \) + 1 values, so degeneracy = 53
1.b) \( P_x \) is directed along \( x \)-axis and so must have \( \sin \Theta \cos \phi \) dependence.

\( P_y \) must have \( \sin \Theta \sin \phi \) dependence.

Look for linear combinations of \( Y_{l+1} \) that produce:

\[
e^{i\phi} + e^{-i\phi} = 2 \cos \phi \quad \text{and} \quad e^{i\phi} - e^{-i\phi} = 2i \sin \phi
\]

\[-Y_{l+1} + Y_{l-1} \quad \text{and} \quad -Y_{l+1} - Y_{l-1}\]

\( P_x \) has the shape: \(-Y_{l+1} + Y_{l-1} = 2 \ N_{l,1} \sin \Theta \cos \phi \)

\( P_y \) has the shape: \(-Y_{l+1} - Y_{l-1} = 2i \ N_{l,1} \sin \Theta \sin \phi \)

\[\text{important thing is to see these linear combinations of } Y_{l \pm 1} \text{ and these sin and cos expressions} \]

1.c) only need to grade the First part.

\[
[\hat{L}^2, \hat{L}_z] = \hat{L}^2 \hat{L}_z - \hat{L}_z \hat{L}^2 = 0
\]

\[
\hat{L}^2 = \hbar^2 \left[ \frac{1}{\sin \Theta} \frac{\partial}{\partial \Theta} \left( \sin \Theta \frac{\partial}{\partial \Theta} \right) + \frac{1}{\sin^2 \Theta} \frac{\partial^2}{\partial \phi^2} \right]
\]

\[
\hat{L}_z = -i \hbar \frac{\partial}{\partial \phi}
\]
Show Commutator is zero by evaluating both terms and equating.

\[ \hat{L}_z^2 \hat{L}_z = \hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] (-i \hbar \partial \phi) \]

\[ = -i \hbar^3 \left[ \frac{1}{\sin \theta} \left( \cos \theta \frac{\partial}{\partial \theta} + \sin \theta \frac{\partial^2}{\partial \phi^2} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \partial \phi \]

\[ = -i \hbar^3 \left[ \cot \theta \frac{\partial}{\partial \theta} \partial \phi + \frac{\partial^2}{\partial \phi^2} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \]

\[ \hat{L}_z \hat{L}_z^2 = -i \hbar \frac{\partial}{\partial \phi} \left[ \hbar^2 \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) \right] \]

\[ = -i \hbar^3 \frac{\partial}{\partial \phi} \left[ \cot \theta \frac{\partial}{\partial \theta} + \frac{\partial^2}{\partial \phi^2} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] = \hat{L}_z \hat{L}_z^2 \ldots \]

because order of differentiation not important

\[ \frac{\partial^2}{\partial \phi \partial \theta} = \frac{\partial}{\partial \theta} \frac{\partial}{\partial \phi} \quad \text{and} \quad \frac{\partial^3}{\partial \phi \partial \theta^2} = \frac{\partial}{\partial \theta} \frac{\partial}{\partial \phi^2} \]

donot need to grad anything else.

importantly, one can see that \[ [\hat{L}_z^2, \hat{L}_x] \] and \[ [\hat{L}_z, \hat{L}_x] \]
will also equal zero because these terms simply bring in more linear combinations of derivatives. Since the order of differentiation is not important, they will all commute with \[ \hat{L}_z \ldots \].
1. \( I \omega = L = \sqrt{l(l+1)} \hbar \)

\[
L = (1.80 \cdot 10^{-4} \text{ kg m}^2)(314 \text{ rad/s}) = 0.0565 \text{ kg m}^2\text{s}
\]

\[
\ell(l+1) = \left(\frac{L}{\hbar}\right)^2 = \left(\frac{2\pi (0.0565 \text{ kg m}^2)}{6.626 \cdot 10^{-34} \text{ kg m}^2\text{s}}\right)^2 = 2.87 \cdot 10^{65}
\]

\( \ell \) is extremely large so \( \ell(l+1) \sim \ell^2 \)

\[
\ell = \sqrt{2.87 \cdot 10^{65}} = 5.36 \cdot 10^{32}
\]

1.e) angular velocity is \( \omega \).

\[
\omega = \frac{L}{I} = \frac{\sqrt{l(l+1)} \hbar}{I}
\]

let \( l = 6 \)

\[
\omega = \frac{\sqrt{62} \left(\frac{1}{2\pi}\right) (6.626 \cdot 10^{-34} \text{ kg m}^2\text{s})}{1.80 \cdot 10^{-4} \text{ kg m}^2}
\]

\[
\omega = 3.80 \cdot 10^{-30} \text{ rad/s}
\]

It would require \( 1.65 \cdot 10^{30} \text{s} \) to make one revolution!

\[= 52.4 \text{ sextillion years}\]

Obviously not perceptible
2.a) Ionization energy = $E_{n=\infty} - E_{n=3}$

$E = -\hbar c K H \frac{Z^2}{n^2} = -13.6 \frac{Z^2}{n^2}$ (eV)

I.E. = 0 + 13.6 $(\frac{3^2}{3^2})$ = 13.6 eV same as H-atom from n = 1

2.b) $V' = -\frac{Z e^2}{4 \pi \varepsilon_0 r} + \frac{l(l+1) \hbar^2}{2 \mu r^2}$

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coulombic attraction due to rotational motion of the electron.

2.c) $\Psi_{1,0,0} = R_{1,0} Y_{0,0} = 2 \left(\frac{Z}{a}\right)^{\frac{3}{2}} e^{-\frac{Z}{4a}} \left(\frac{1}{4\pi}\right)^{\frac{3}{2}}$

$\Psi_{2,1,0} = R_{2,1} Y_{1,0} = \frac{1}{\sqrt{24}} \left(\frac{Z}{a}\right)^{\frac{3}{2}} \rho e^{-\frac{Z}{4a}} \left(\frac{3}{4\pi}\right)^{\frac{3}{2}} \cos \theta$

$\Psi_{3,2,0} = R_{3,2} Y_{2,0} = \frac{1}{\sqrt{2430}} \left(\frac{Z}{a}\right)^{\frac{3}{2}} \rho^2 e^{-\frac{Z}{4a}} \left(\frac{5}{16\pi}\right)^{\frac{3}{2}} (3 \cos^2 \theta - 1)$

must define $\rho$ for each as it changes with $n.$
2.d) \[ R_{1,0} = 2 \left( \frac{2}{a} \right)^{3/2} e^{-\rho^2/a} \]

2.e) \[ Y_{0,0} = \left( \frac{1}{4\pi} \right)^{1/2} \]

spherical

\[ Y_{1,0} = \left( \frac{3}{4\pi} \right)^{1/2} \cos \theta \]

Find maximum amplitudes \( \theta = 0, \pi \Rightarrow z\text{-axis} \)

all \( \phi \) the same

\[ Y_{2,0} = \left( \frac{5}{16\pi} \right)^{1/2} (3\cos^2 \theta - 1) \]

Find maximum amplitudes

\( \theta = 0, \pi, \) also is large negative at \( \theta = \frac{\pi}{2} \)

\( \) and all \( \phi \) the same

\( Y_{2,0} = 0 \) where

\( 3\cos^2 \theta = 1 \)

\( \Theta = 54.7^\circ \)

\( = 144.7^\circ \)

choose to show XZ axes

\[ Y_{2,0} = 0 \text{ at:} \]

\( \theta = \frac{\pi}{2} \)

xy plane